

Fractional Set Systems with Few Disjoint Pairs

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Abstract

Ahlsweide and Frankl independently found a result about the structure of set systems with few disjoint pairs. Bollobás and Leader gave an alternate proof by generalizing to fractional set systems and noting that the optimal fractional set systems are $\{0, 1\}$ -valued. In this paper we show that this technique does not extend to t -intersecting families. We find optimal fractional set systems for some infinite classes of parameters, and we point out that they are strictly better than the corresponding $\{0, 1\}$ -valued fractional set systems. We prove some results about the structure of an optimal fractional set system, which we use to produce an algorithm for finding such systems. The run-time of the algorithm is polynomial in the size of the ground set.

Keywords: Intersecting Family; Fractional Set System; Extremal Problem; Hypergraph

1. Introduction

Mathematicians have long been interested in intersecting set systems. A collection of sets $\mathcal{S}$ is *intersecting* if every pair of sets in the collection has non-empty intersection. Suppose that every element of $\mathcal{S}$ is a subset of $[n] = \{1, 2, \dots, n\}$. In that case we say that $\mathcal{S}$ is a *family on the ground set* $[n]$. It is well-known and easy to show that an intersecting family on the ground set $[n]$ has at most 2^{n-1} elements.

A family $\mathcal{S}$ is *t -intersecting* if $|A \cap B| \geq t$ for each pair of sets $A, B \in \mathcal{S}$. The maximum size of a t -intersecting family is given by a theorem of Katona

[1]. Denote this maximum by $M(n, t)$. That is,

$$M(n, t) = \max \left\{ |\mathcal{A}| : \mathcal{A} \subseteq 2^{[n]}, \mathcal{A} \text{ is } t\text{-inteseacting} \right\},$$

where $2^{[n]} = \{S : S \subseteq [n]\}$ is the powerset of $[n]$. Katona's result is the following.

Theorem 1 (Katona). *The largest t -intersecting family on $[n]$ consists of all of the sets of size at least $\frac{(n+t)}{2}$ and some of size $\frac{n+t-1}{2}$. More precisely,*

$$M(n, t) = \begin{cases} \binom{\frac{n+t}{2}}{>}, & \text{if } (n+t) \text{ is even;} \\ 2\binom{\frac{n-1}{2}}{\geq \frac{n+t-1}{2}}, & \text{if } (n+t) \text{ is odd,} \end{cases}$$

where

$$\binom{n}{\geq r} = \sum_{i=r}^n \binom{n}{i}.$$

In addition to t -intersecting families, it is also natural to consider t -intersecting hypergraphs. That is, given fixed $n, r \in \mathbb{N}$, we ask for the largest t -intersecting subset of $\binom{[n]}{r}$ (the size r subsets of $[n]$). The $t = 1$ case is settled by the well-known Erdős-Ko-Rado Theorem [2]. Erdős, Ko, and Rado also establish the solution for $t > 1$ and n large enough. The complete answer was settled by Ahlswede and Khachatrian [3].

A natural next question is: given (large) $s \in \mathbb{N}$, how close to t -intersecting can a family of size s be? More precisely, given $\mathcal{A} \subseteq 2^{[n]}$, define $D_t(\mathcal{A})$ to be the number of pairs $(A, B) \in \mathcal{A} \times \mathcal{A}$ so that $|A \cap B| < t$. Given $s \in \mathbb{N}$, we wish to minimize $D_t(\mathcal{A})$ over all systems with $|\mathcal{A}| = s$. Call this minimum $D_{n,t}(s)$. The theorem of Katona given above establishes the values of s for which $D_{n,t}(s) = 0$.

Frankl [4] and Ahlswede [5] independently determined the answer for particular values of s when $t = 1$. Essentially, the optimal family has as many large sets as possible. Given $n, r \in \mathbb{N}$, denote $\binom{[n]}{\geq r} = \{A \subseteq [n] : |A| \geq r\}$. Such a family is a *discrete Hamming ball*. We have the following.

Theorem 2 (Frankl, Ahlswede). *There is an optimal set system between discrete adjacent Hamming balls. More precisely: given $\mathcal{B} \subseteq 2^{[n]}$, let r be such that $\binom{n}{\geq r+1} \leq |\mathcal{B}| \leq \binom{n}{\geq r}$. Then there is $\mathcal{A} \subseteq 2^{[n]}$ with $|\mathcal{A}| = |\mathcal{B}|$, $\binom{[n]}{\geq r+1} \subseteq \mathcal{A} \subseteq \binom{[n]}{\geq r}$ and $D_1(\mathcal{A}) \leq D_1(\mathcal{B})$.*

An immediate corollary of this theorem is that if $s = \binom{n}{\geq r}$, then $D_1(s) = D_1(\binom{[n]}{\geq r})$. Bollobás and Leader [6] give another proof of this corollary (but not of the theorem) by generalizing to fractional set systems and noting that the optimal fractional set systems are $\{0, 1\}$ -valued.

One would like to apply this technique for general t . However, the theorem of Bollobás and Leader is false for $t > 1$. In section 4 we give some (simple) counter examples. The question of determining the D_t minimizing fractional set systems is still open. We give some results about the structure of an optimal fractional set system. We provide an algorithm, polynomial in the size of the ground set, to determine this fractional set system.

2. The Theorem of Bollobás and Leader

Given $n \in \mathbb{N}$, we define a *fractional set system on $[n]$* to be a map

$$f : 2^{[n]} \rightarrow [0, 1].$$

The $\{0, 1\}$ -valued fractional systems correspond to classical set systems. We denote the set of all fractional set systems on $2^{[n]}$ by $\mathcal{F}_n$. If $f \in \mathcal{F}_n$, we define its *weight*, $W(f)$, by

$$W(f) = \sum_{A \in 2^{[n]}} f(A).$$

Given $r, s \in \mathbb{R}$, define

$$r \oplus s = \max \{0, a + b - 1\}.$$

One might think of $r \oplus s$ is the volume of liquid that spills out of a test tube of volume 1 if liquids of volume r and s are added to it. Given $t \in \mathbb{N}$, we define

$$D_t(f) = \sum_{\substack{(A, B) \in 2^{[n]} \times 2^{[n]} \\ |A \cap B| < t}} f(A) \oplus f(B).$$

Notice that this definition of D_t extends the previous definition. Given a fixed weight $w \geq 0$, we want to find

$$D_{n,t}(w) = \inf \{D_t(f) : f \in \mathcal{F}_n, W(f) = w\}$$

To apply induction, it is useful to count the number of disjoint pairs between two (often different) fractional set systems. Given $n, t \in \mathbb{N}$ and $f, g \in \mathcal{F}_n$ we define

$$D_t(f, g) = \sum_{\substack{(A, B) \in 2^{[n]} \times 2^{[n]} \\ |A \cap B| < t}} f(A) \oplus g(B).$$

Given $v, w \in \mathbb{R}$, define

$$D_{n,t}(v, w) = \inf \{ D_t(f, g) : f, g \in \mathcal{F}_n, W(f) = v, W(g) = w \}.$$

Notice in particular that $D_t(f) = D_t(f, f)$.

Given a fixed weight w with $0 \leq w \leq 2^n$, there is exactly one $f \in \mathcal{F}_n$ of weight w for which there exists $k \in [0, n]$ and $\alpha \in [0, 1]$ such that

$$f(A) = \begin{cases} 1, & |A| > k; \\ \alpha, & |A| = k; \\ 0, & |A| < k. \end{cases}$$

Following Bollobás and Leader, we call this the *Hamming ball of weight w on $2^{[n]}$* , and denote it by b_n^w .

Bollobás and Leader proved the following theorem.

Theorem 3 (Bollobás and Leader [6]). *When $t = 1$ the optimal fractional set system is a Hamming ball. That is, for $n \in \mathbb{N}$ and $v, w \in \mathbb{R}$,*

$$D_{n,1}(v, w) = D_1(b_n^v, b_n^w).$$

Is this theorem true for $t > 1$? We give a very small counter example to establish that it is not.

Example 4. *Let $t > 1$ and consider $D_{1,t}(1, 1)$. Let $f \in \mathcal{F}_1$ be given by $f(\{1\}) = .5$ and $f(\emptyset) = .5$. We have that $D_t(f, f) = 0$. On the other hand, $D_t(b_1^1, b_1^1) = 1$. Similarly, for any positive integers n, t with $2 \leq t \leq n$, $D_{n,t}(2^{n-1}, 2^{n-1}) = 0$, but $D_t(b_n^{2^{n-1}}, b_n^{2^{n-1}}) > 0$.*

It is not the case that there are only counter examples for relatively low weights. To see this, we first establish some general facts about D_t minimizing fractional set systems, and then use these to give a large class of counter examples, and finally to produce an efficient algorithm to “graph” $D_{n,t}(w)$ for given n and t .

3. Optimal Fractional Set Systems

Let $0 \leq w \leq 2^n$. By a compactness argument, there is $f \in \mathcal{F}_n$ with $W(f) = w$ and $D_{n,t}(w) = D_t(f)$. We now prove some facts about the structure of optimal fractional set systems.

We call a fractional set system $f \in \mathcal{F}_n$ *constant on layers* if for all $A, B \in 2^{[n]}$, $|A| = |B|$ implies $f(A) = f(B)$. We may turn any fractional set system into one which is constant on layers; given $f \in \mathcal{F}_n$, we define the *smear operation* $\sigma : \mathcal{F}_n \rightarrow \mathcal{F}_n$ by

$$\sigma(f)(A) = \binom{n}{|A|}^{-1} \sum_{B \in \binom{[n]}{|A|}} f(B).$$

Lemma 5. *Given fractional set systems $f, g \in \mathcal{F}_n$, $D_t(\sigma(f), \sigma(g)) \leq D_t(f, g)$.*

Thus for any weight, there is a D_t -minimizing fractional set system that is constant on layers. To prove lemma 5, we establish a more general fact that relies on the convexity of D_t , which we now prove.

Lemma 6. *The function D_t is convex. That is, given $n, t \in \mathbb{N}$, $f_1, g_1, f_2, g_2 \in \mathcal{F}_n$ and $\lambda \in [0, 1]$, we have*

$$D_t(\lambda(f_1, g_1) + (1 - \lambda)(f_2, g_2)) \leq \lambda D_t(f_1, g_1) + (1 - \lambda) D_t(f_2, g_2).$$

Proof. Notice that the function $h(x) = \max\{0, 1 - x\}$ is convex. This justifies the only inequality below.

$$\begin{aligned} & D_t(\lambda(f_1, g_1) + (1 - \lambda)(f_2, g_2)) \\ &= \sum_{\substack{(A, B) \in 2^{[n]} \times 2^{[n]} \\ |A \cap B| < t}} (\lambda(f_1(A)) + (1 - \lambda)f_2(A)) \oplus (\lambda(g_1(A)) + (1 - \lambda)g_2(A)) \\ &= \sum_{\substack{(A, B) \in 2^{[n]} \times 2^{[n]} \\ |A \cap B| < t}} h(\lambda(f_1(A) + g_1(B) - 1) + (1 - \lambda)(f_2(A) + g_2(B) - 1)) \\ &\leq \sum_{\substack{(A, B) \in 2^{[n]} \times 2^{[n]} \\ |A \cap B| < t}} \lambda h(f_1(A) + g_1(B) - 1) \\ &\quad + \sum_{\substack{(A, B) \in 2^{[n]} \times 2^{[n]} \\ |A \cap B| < t}} (1 - \lambda) h(f_2(A) + g_2(B) - 1) \\ &= \lambda D_t(f_1, g_1) + (1 - \lambda) D_t(f_2, g_2) \end{aligned}$$

□

Define the graph $G_{n,t}$ to be the bipartite graph each of whose partite sets is a copy of $2^{[n]}$, and where AB is an edge if $|A \cap B| < t$. Let $\mathcal{P}$ be a partition of the vertices of $G_{n,t}$. Given $v \in G_{n,t}$, let P_v be the part that contains v . A pair $(f, g) \in \mathcal{F}_n \times \mathcal{F}_n$ is naturally a vertex weighting $(f, g) : V(G_{n,t}) \rightarrow \mathbb{R}$. Define $\sigma_{\mathcal{P}}(f, g)$ to be the pair $(f', g') \in \mathcal{F}_n \times \mathcal{F}_n$ given by

$$(f', g')(v) = \frac{1}{|P_v|} \sum_{v' \in P_v} (f, g)(v').$$

In other words, $\sigma_{\mathcal{P}}$ averages the parts of $\mathcal{P}$. In particular, if $\mathcal{A}$ is a group of automorphisms of $G_{n,t}$, we define $\sigma_{\mathcal{A}} = \sigma_{\mathcal{O}}$, where $\mathcal{O}$ is the set of orbits of $\mathcal{A}$.

Lemma 7. *Let $n, t \in \mathbb{N}$ and let $\mathcal{A}$ be a group of automorphisms of $G_{n,t}$. Then given $f, g \in \mathcal{F}_n$,*

$$D_t(\sigma_{\mathcal{A}}(f, g)) \leq D_t(f, g).$$

Proof. Since $\mathcal{A}$ acts on $V(G_{n,t})$, there is a natural action of $\mathcal{A}$ on $\mathcal{F}_n \times \mathcal{F}_n$; given $\phi \in \mathcal{A}$ and $(f, g) \in \mathcal{F}_n \times \mathcal{F}_n$, we define $\phi(f, g)$ to be the function in $\mathcal{F}_n \times \mathcal{F}_n$ given by

$$\phi(f, g)(v) = (f, g)(\phi^{-1}(v)).$$

Notice that

$$D_t(\phi(f, g)) = D_t(f, g)$$

and

$$\sigma_{\mathcal{A}}(f, g) = \frac{1}{|\mathcal{A}|} \sum_{\phi \in \mathcal{A}} \phi(f, g),$$

and so by the convexity of D_t and Jensen's inequality,

$$D_t(\sigma_{\mathcal{A}}(f, g)) \leq \frac{1}{|\mathcal{A}|} \sum_{\phi \in \mathcal{A}} D_t(\phi(f, g)) = D_t(f, g).$$

□

We may use this fact to easily establish lemma 5.

Proof of lemma 5. Let $\phi \in S_n$ be a permutation of $[n]$. Notice that ϕ induces a graph automorphism on $G_{n,t}$: a vertex $A \subseteq [n]$ in a part of $G_{n,t}$ is sent to the copy of $\phi(A)$ in the same part. Two vertices $v, w \in G_{n,t}$ are in the same orbit of S_n if and only if $|v| = |w|$ and v and w are in the same partite set. Clearly then $\sigma = \sigma_{\mathcal{A}}$, and so the claim follows by Lemma 7. □

Notice that in $G_{n,t}$, every set $A \in \binom{[n]}{< t}$ is connected to every other set (in the opposite part). Thus the set of maps that permute these vertices (without

changing partitions) and fix all other vertices is a group of automorphisms of $G_{n,t}$. By applying Lemma 7 we then have the following fact.

Lemma 8. *Given $n \in \mathbb{N}$, $t \in [0, n]$, and $u, w \in \mathbb{R}$, with $0 \leq u, w \leq 2^{[n]}$, there are $f, g \in \sigma(\mathcal{F}_n)$ with $W(f) = u$, and $W(g) = w$ such that $D_{n,t}(u, w) = D_t(f, g)$ and for all $A, B \in \binom{[n]}{<_t}$, $f(A) = f(B)$ and $g(A) = g(B)$. $\square$*

We call such a fractional set system *constant below t* . From now on we may assume without loss of generality that a D_t -minimizing function f is constant on layers and constant below t . We call such a fractional set system *t -constant*, and denote the set of t -constant fractional set systems of weight w on $2^{[n]}$ by $\mathcal{K}_{n,t}^w$. Given $f \in \mathcal{K}_{n,t}^w$, and $i \in [0, n]$ we may define

$$f_i = f(A),$$

where A is any set in $\binom{[n]}{i}$. For all $j, k < t$, we have $f_j = f_k$. We will denote this common weight by $f_{<t}$.

Given $n, t \in \mathbb{N}$, and $w \in \mathbb{R}$, with $0 \leq w \leq 2^n$, we would like there to exist an $f \in \mathcal{K}_{n,t}^w$ such that $D_{n,t}(w, w) = D_t(f, f)$. This would imply $D_{n,t}(w, w) = D_{n,t}(w)$. The following lemma establishes this.

Lemma 9. *Given $n, t \in \mathbb{N}$ and $f, g \in \mathcal{F}_n$, $D_t\left(\frac{f+g}{2}, \frac{f+g}{2}\right) \leq D_t(f, g)$.*

Proof. Notice that the function $\phi : G_{n,t} \rightarrow G_{n,t}$ that switches the two parts is a graph automorphism. Further, $\mathcal{A} = \{1, \phi\}$ is a group and applying Lemma 7 gives the desired result. $\square$

We call a function $f \in \mathcal{F}_n$ *nondecreasing* if for all $A, B \in 2^{[n]}$ with $|A| \leq |B|$ we have $f(A) \leq f(B)$. (Notice that if f is nondecreasing then it is constant on layers.) We have the following lemma.

Lemma 10. *Given $n \in \mathbb{N}$ and $v, w \in \mathbb{R}$, there are nondecreasing set systems $f, g \in \mathcal{F}_n$ with $W(f) = v$ and $W(g) = w$ such that $D_{n,t}(v, w) = D_t(f, g)$.*

To aid in the proof we introduce some notation. Given $n, t \in \mathbb{N}$, $A \in 2^{[n]}$, and $j \in [0, n]$, the number of j -sets of $[n]$ that are *t -disjoint* from A is

$$\Lambda_{n,t}(A, j) = \Lambda(A, j) = \left| \left\{ B \in \binom{[n]}{j} : |A \cap B| < t \right\} \right|.$$

Notice that as a function of A , $\Lambda_{n,t}(A, j)$ only depends on $|A|$. In particular, if $|A| = i$, we define

$$\Lambda(i, j) = \Lambda(A, j) = \sum_{d=0}^{t-1} \binom{i}{d} \binom{n-i}{j-d}.$$

Proof of Lemma 10. Let $n, t \in \mathbb{N}$ and $v, w \in \mathbb{R}$. Let $f \in \mathcal{K}_{n,t}^v$ and $g \in \mathcal{K}_{n,t}^w$ have $D_{n,t}(v, w) = D_t(f, g)$. We may assume (by another compactness argument) that $s(f, g) = \sum_{i=0}^n i(f_i + g_i)$ is maximized over all pairs $(f, g) \in \mathcal{K}_{n,t}^v \times \mathcal{K}_{n,t}^w$ that satisfy $D_t(f, g) = D_{n,t}(v, w)$. We want to show that f and g are nondecreasing. Suppose by way of contradiction that there are integers $0 \leq i < j \leq n$ such that $f_i > f_j$ (the case $g_i > g_j$ is the same). We will shift some weight from i to j obtaining f' with $s(f', g) > s(f, g)$ and yet $D_t(f', g) \leq D_t(f, g)$, a contradiction. To that end, choose $\Delta W > 0$ small enough that for all $k \in [0, n]$ for which $f_j + f_k < 1$, we have $f_j + f_k + \binom{n}{j}^{-1} \Delta W < 1$. The quantity ΔW should also be small enough that $f_i - \binom{n}{i}^{-1} \Delta W \geq f_j + \binom{n}{j}^{-1} \Delta W$. Finally, it should be the case that $f_i - \binom{n}{i}^{-1} \Delta W \geq 0$ and $f_j + \binom{n}{j}^{-1} \Delta W \leq 1$. Define $f' \in \mathcal{F}_n$ by

$$f'(A) = \begin{cases} f(A), & |A| \notin \{i, j\}; \\ f_j + \binom{n}{j}^{-1} \Delta W & |A| = j; \\ f_i - \binom{n}{i}^{-1} \Delta W & |A| = i. \end{cases}$$

Notice that $W(f') = W(f)$, and that $s(f', g) = s(f, g) + (j - i)\Delta W$. Further notice that

$$\begin{aligned} D_t(f, g) - D_t(f', g) &= \sum_{\substack{A \in \binom{[n]}{j}, B \in 2^{[n]} \\ |A \cap B| < t}} [f(A) \oplus g(B) - f'(A) \oplus g(B)] \\ &\quad + \sum_{\substack{A \in \binom{[n]}{i}, B \in 2^{[n]} \\ |A \cap B| < t}} [f(A) \oplus g(B) - f'(A) \oplus g(B)] \\ &= \sum_{k=0}^n \left(\sum_{\substack{A \in \binom{[n]}{j}, B \in \binom{[n]}{k} \\ |A \cap B| < t}} [f(A) \oplus g(B) - f'(A) \oplus g(B)] \right. \\ &\quad \left. + \sum_{\substack{A \in \binom{[n]}{i}, B \in \binom{[n]}{k} \\ |A \cap B| < t}} [f(A) \oplus g(B) - f'(A) \oplus g(B)] \right) \end{aligned}$$

Fix $k \in [0, n]$. We show that the corresponding term in the last sum is non-negative. We have two cases: either $f_j + f_k < 1$ or $f_j + f_k \geq 1$. If $f_j + f_k < 1$ then by our choice of ΔW , we have $f'_j + f_k < 1$. Thus for all $A \in \binom{[n]}{j}$ and $B \in \binom{[n]}{k}$, $f(A) \oplus g(B) = 0$ and $f'(A) \oplus g(B) = 0$. For $A \in \binom{[n]}{i}$ and $B \in \binom{[n]}{k}$,

$f(A) \oplus g(B) - f'(A) \oplus g(B)$ is always nonnegative, so the case $f_j + f_k < 1$ is settled.

If $f_j + f_k \geq 1$, then $f_i + f_k \geq 1$, $f'_j + f_k \geq 1$, and $f'_i + f_k \geq 1$ by the choice of ΔW . Thus

$$\begin{aligned}
& \sum_{\substack{A \in \binom{[n]}{j}, B \in \binom{[n]}{k} \\ |A \cap B| < t}} [f(A) \oplus g(B) - f'(A) \oplus g(B)] \\
&= \sum_{\substack{A \in \binom{[n]}{j}, B \in \binom{[n]}{k} \\ |A \cap B| < t}} \left[(f_j + g_k - 1) - (f_j + \binom{n}{j}^{-1} \Delta W + g_k - 1) \right] \\
&= - \binom{[n]}{j} \Lambda(j, k) \binom{n}{j}^{-1} \Delta W \\
&= -\Lambda(j, k) \Delta W
\end{aligned}$$

Similarly,

$$\sum_{\substack{A \in \binom{[n]}{i}, B \in \binom{[n]}{k} \\ |A \cap B| < t}} [f(A) \oplus g(B) - f'(A) \oplus g(B)] = \Lambda(i, k) \Delta W$$

Notice that because $j > i$, there are more k sets that have small intersection with a given i set than there are k sets that have small intersection with a given j set. In other words, $\Lambda(j, k) \leq \Lambda(i, k)$. Hence the term corresponding to k in the sum above is nonnegative. $\square$

We say a set system $f \in \mathcal{K}_{n,t}^w$ is *t-canonical* if f is nondecreasing. We denote the set of *t*-canonical fractional set systems of weight w by $\mathcal{C}_{n,t}^w$.

4. More Counter Examples

In this section we give some more counter examples to the $t > 1$ case of the theorem of Bollobás and Leader, and we find the minimizing fractional set system when $t > \lceil \frac{n}{2} \rceil$.

Example 11. Let $n, t \in \mathbb{N}$, $t \leq n$. For $w \in \mathbb{R}$ with

$$\binom{n}{\geq t} + 1/2 \binom{n}{t-1} < w < 2^n - 1/2,$$

We have $D_t(b_n^w) > D_{n,t}(w)$.

Proof. Let $A = \{1, 2, \dots, t-1\}$. By our choice of w , $b_n^w(A) > 1/2$. Let $\Delta w > 0$ be less than $b_n^w(A) - 1/2$ and less than $1/2 - b_n^w(\emptyset)$. Consider $f \in \mathcal{F}_n$ defined by

$$f(B) = \begin{cases} b_n^w(\emptyset) + \Delta w & \text{if } B = \emptyset \\ b_n^w(A) - \Delta w & \text{if } B = A \\ b_n^w(B) & \text{if } B \notin \{A, \emptyset\}. \end{cases}$$

Since A and $\emptyset$ are t -disjoint from everything, we have that

$$\begin{aligned} D_t(b_n^w) - D_t(f) &= 2 \sum_{B \in 2^{[n]} \setminus \{\emptyset, A\}} [b_n^w(A) \oplus b_n^w(B) + b_n^w(\emptyset) \oplus b_n^w(B)] \\ &\quad - 2 \sum_{B \in 2^{[n]} \setminus \{\emptyset, A\}} [f(A) \oplus f(B) + f(\emptyset) \oplus f(B)] \\ &\quad + b_n^w(A) \oplus b_n^w(A) + b_n^w(\emptyset) \oplus b_n^w(\emptyset) \\ &\quad - f(A) \oplus f(A) - f(\emptyset) \oplus f(\emptyset) \\ &= 2 \sum_{B \in 2^{[n]} \setminus \{\emptyset, A\}} [(b_n^w(A) \oplus b_n^w(B) - f(A) \oplus f(B))] \\ &\quad + 2 \sum_{B \in 2^{[n]} \setminus \{\emptyset, A\}} [(b_n^w(\emptyset) \oplus b_n^w(B) - f(\emptyset) \oplus f(B))] \\ &\quad + (b_n^w(A) + b_n^w(A) - 1) - (f(A) + f(A) - 1) \\ &= 2 \sum_{B \in 2^{[n]} \setminus \{\emptyset, A\}} [(b_n^w(A) \oplus b_n^w(B) - f(A) \oplus f(B))] \\ &\quad + 2 \sum_{B \in 2^{[n]} \setminus \{\emptyset, A\}} [(b_n^w(\emptyset) \oplus b_n^w(B) - f(\emptyset) \oplus f(B))] \\ &\quad + 2\Delta w \end{aligned}$$

Notice that $b_n^w(A) \oplus b_n^w(B) - f(A) \oplus f(B) \geq 0$, and furthermore, if $b_n^w(\emptyset) \oplus b_n^w(B) - f(\emptyset) \oplus f(B) < 0$, then $f(\emptyset) + f(B) > 1$, so $f(B) > 1/2$ and thus since $b_n^w(B) = f(B)$, we have

$$\begin{aligned} &(b_n^w(A) \oplus b_n^w(B) - f(A) \oplus f(B)) + (b_n^w(\emptyset) \oplus b_n^w(B) - f(\emptyset) \oplus f(B)) \\ &= (f(A) + \Delta w + f(B) - 1) - (f(A) + f(B) - 1) \\ &\quad + (f(\emptyset) - \Delta w) \oplus f(B) - (f(\emptyset) + f(B) - 1) \\ &\geq \Delta w \\ &\quad + (f(\emptyset) - \Delta w + f(B) - 1) - (f(\emptyset) + f(B) - 1) \\ &= 0 \end{aligned}$$

Thus $D_t(b_n^w) - D_t(f) \geq 2\Delta w > 0$ as desired. $\square$

Given $n, t \in \mathbb{N}$, and a fixed weight w with $2^{n-1} \leq w \leq 2^n$, there is exactly one $f \in \mathcal{C}_{n,t}^w$ of the form

$$f(A) = \begin{cases} 1, & |A| > k; \\ \alpha, & |A| = k; \\ 1/2, & |A| < k. \end{cases}$$

where $k \in [t-1, n]$ and $\alpha \in \mathbb{R}$ with $1/2 \leq \alpha \leq 1$. We call this the t -half-ball of weight w on $2^{[n]}$. For $w < 2^{n-1}$ the system with constant weight $f(A) = \frac{w}{2^n}$ will also be called a t -half-ball. We denote the t -half-ball of weight w on $2^{[n]}$ by $h_{n,t}^w$.

Theorem 12. Let $n, t \in \mathbb{N}$ with $t > \left\lfloor \frac{n}{2} \right\rfloor$, and let $w \in \mathbb{R}$ with $0 \leq w \leq 2^n$. Then among the optimal fractional set systems is the t -half-ball. That is,

$$D_{n,t}(w) = D_t(h_{n,t}^w).$$

Proof. If $w \leq 2^{n-1}$, then $h_{n,t}^w$ is the constant fractional set system with total weight w . This constant value is no more than $1/2$, so $D_t(h_{n,t}^w) = 0 = D_{n,t}(w)$ as desired. Thus we assume that $w > 2^{n-1}$.

Now let $f \in \mathcal{C}_{n,t}^w$ with $D_{n,t}(w) = D_t(f)$. We may assume (by a compactness argument) that $f_{<t}$ is as large as possible. We claim that $f_{<t} \geq 1/2$. Notice that in these circumstances,

$$D_t(f) = \sum_{\substack{(A,B) \in 2^{[n]} \times 2^{[n]} \\ |A \cap B| < t}} f(A) + f(B) - 1,$$

and thus we want to keep weight in the sets that occur least often in the sum. That is, we want as much weight as possible in large sets while maintaining the property that the weight on every set is at least $1/2$. Of course, $h_{n,t}^w$ does exactly that, and so $D_{n,t}(w) = D_t(f) \geq D_t(h_{n,t}^w)$, and the theorem is proved.

Suppose $f_{<t} < 1/2$. Let $l = \min \{j \in [0, n] : f_j \geq 1/2\}$. (Notice that $\{j \in [0, n] : f_j \geq 1/2\}$ is nonempty since $w > 2^{n-1}$.) Choose $\Delta w > 0$ small enough so that $f_{l-1} + \binom{n}{\leq l-1}^{-1} \Delta w \leq f_l - \binom{n}{\geq l}^{-1} \Delta w$ and $f_{l-1} + \binom{n}{\leq l-1}^{-1} \Delta w \leq 1/2$. Define $g \in \mathcal{C}_{n,t}^w$ by

$$g_i = \begin{cases} f_i - \binom{n}{\geq l}^{-1} \Delta w & \text{if } i \geq l; \\ f_i + \binom{n}{\leq l-1}^{-1} \Delta w & \text{if } i < l. \end{cases}$$

We show that for any $(i, j) \in [0, n] \times [0, n]$, $g_i \oplus g_j \leq f_i \oplus f_j$. For $(i, j) \in [l, n] \times [l, n]$, we have $g_i < f_i$ and $g_j < f_j$. Thus $g_i \oplus g_j \leq f_i \oplus f_j$. For $(i, j) \in [l, n] \times [0, l-1]$, we have $g_i + g_j = f_i - \binom{n}{\geq l}^{-1} \Delta w + f_j + \binom{n}{\leq l-1}^{-1} \Delta w$. But

$l \geq t > \lceil n/2 \rceil$, so $\binom{n}{\geq l} \leq \binom{n}{\leq l-1}$ and hence $g_i + g_j \leq f_i + f_j$, so $g_i \oplus g_j \leq f_i \oplus f_j$. Similarly, for $(i, j) \in [0, l-1] \times [0, n]$, $g_i \oplus g_j \leq f_i \oplus f_j$. If $(i, j) \in [0, l-1] \times [0, l-1]$, then by our choice of Δw , g_i and g_j are both no more than $1/2$, and so $g_i + g_j \leq 1$, and $g_i \oplus g_j = 0 \leq f_i \oplus f_j$.

Now we easily have

$$\begin{aligned} D_t(g) &= \sum_{(i,j) \in [0,n] \times [0,n]} \binom{n}{i} \Lambda_{i,j}(g_i \oplus g_j) \\ &\leq \sum_{(i,j) \in [0,n] \times [0,n]} \binom{n}{i} \Lambda_{i,j}(f_i \oplus f_j) \\ &= D_t(f) \end{aligned}$$

But by our choice of Δw , $g \in \mathcal{C}_{n,t}^w$ and furthermore $g_{<t} > f_{<t}$. This is a contradiction. Hence $f_{<t} \geq 1/2$, and the theorem follows. $\square$

Corollary 13. *If $n \geq 1$, $t > \lfloor \frac{n}{2} \rfloor$, and $\lfloor \frac{n+t}{2} \rfloor < w < 2^n - 1/2$ then the Hamming ball is not optimal. That is,*

$$D_t(b_n^w) > D_{n,t}(w).$$

Proof. If $w > \binom{n}{\geq t} + 1/2 \binom{n}{t-1}$ then we are done by Example 11. Suppose then that $w \leq \binom{n}{\geq t} + 1/2 \binom{n}{t-1}$. Let $f \in \mathcal{F}_n$ be non-decreasing and constant on layers. Notice that if $f_{t-1} < 1/2$, and $D_t(f) > 0$, then the shift in the proof above strictly decreases D_t . We now show that $D_t(b_n^w) > 0$. Let

$$A = \left\{ 1, 2, \dots, \left\lfloor \frac{n+t}{2} \right\rfloor - 1 \right\},$$

and

$$B = \left\{ n - \left\lfloor \frac{n+t}{2} \right\rfloor + 1, \dots, n \right\}.$$

Then

$$\begin{aligned} |A \cap B| &\leq \left\lfloor \frac{n+t}{2} \right\rfloor - 1 - (n - \left\lfloor \frac{n+t}{2} \right\rfloor + 1) + 1 \\ &= 2 \left\lfloor \frac{n+t}{2} \right\rfloor - n - 1 \\ &\leq t - 1. \end{aligned}$$

But $|A| = \left\lfloor \frac{n+t}{2} \right\rfloor - 1$, and $|B| = \left\lfloor \frac{n+t}{2} \right\rfloor$, and since $w > \binom{n}{\lfloor \frac{n+t}{2} \rfloor}$, $b_n^w(A) > 0$ and $b_n^w(B) = 1$. Hence $b_n^w(A) \oplus b_n^w(B) > 0$ and so $D_t(b_n^w) > 0$. But $w \leq \binom{n}{\geq t} + 1/2 \binom{n}{t-1} < \binom{n}{\geq t} + 1/2 \binom{n}{\leq t-1}$. Thus smearing b_n^w below t yields f , a set system with the property that $f_{<t} < 1/2$, and $D_t(f) \leq D_t(b_n^w)$. If $D_t(f) < D_t(b_n^w)$ we are done. Otherwise $D_t(f) = D_t(b_n^w) > 0$. Applying the shift above, we strictly decrease D_t , establishing the result. $\square$

5. An Algorithmic Solution

Given $n, t \in \mathbb{N}$ and $w \in \mathbb{R}$ with $0 \leq w \leq 2^n$, we say $f \in \mathcal{F}_n$ is a *pseudo-ball* if all of its values are 0, $1/2$, or 1. Notice that the number of pseudo-balls in $\mathcal{C}_{n,t}^w$ is finite. We have the following theorem, which we prove later.

Theorem 14. $D_{n,t}(w)$ is the maximum convex function below the points

$$\{(W(f), D_{n,t}(f)) : f \text{ is a pseudo-ball}\}$$

In particular, $D_{n,t}(w)$ is piecewise linear, and the points where the slope changes correspond to pseudo-balls. Thus we may “graph” $D_{n,t}(w)$ as follows: compute D_t for each of the pseudo-balls, and then use a convex hull algorithm to find $D_{n,t}(w)$. There are $\binom{n-t+4}{2}$ pseudo-balls in $\mathcal{C}_{n,t}^w$, and it takes $O(n^2)$ time to compute D_t of a given pseudo-ball. This yields an $O(n^4)$ run time to compute D_t for every pseudo-ball. We apply the convex hull algorithm to the $O(n^2)$ pseudo-balls. This takes $O(n^4)$ time, and so overall this process completes in $O(n^4)$ time.

Lemma 15. *Given $n, t \in \mathbb{N}$ and $0 \leq w \leq 2^n$, there is $f \in \mathcal{C}_{n,t}^w$ with $D_{n,t}(w) = D_t(f)$ and with $(f_i)_{i=0}^n$ having the form*

$$(f_i)_{i=0}^n = (\underbrace{0, \dots, 0}_{l_0}, \underbrace{1 - \delta, \dots, 1 - \delta}_{l_{1-\delta}}, \underbrace{1/2, \dots, 1/2}_{l_{1/2}}, \underbrace{\delta, \dots, \delta}_{l_\delta}, \underbrace{1, \dots, 1}_{l_1})$$

Where $1/2 < \delta < 1$, and $l_0, l_{1-\delta}, l_{1/2}, l_\delta, l_1 \in [0, n+1]$ are integers that sum to $n+1$.

Proof of Lemma 15. We will see that the space $\mathcal{C}_{n,t}^w$ can be divided into finitely many parts, $P_1, \dots, P_p$, defined by linear inequalities, in such a way that D_t is an affine function on each part. Thus, on each part, minimizing D_t is a linear programming problem. Recall that if a (minimizing) solution to a linear programming problem exists, then there is a solution at a vertex of the feasible region of the problem. We will see that for all i , every vertex in P_i is of the form claimed in the lemma. Since the D_t minimizing f must appear in one of the parts, the lemma is established.

First, we may think of a fractional set system $f \in \mathcal{C}_{n,t}^w$ as a function in $\mathbb{R}^{\{t-1, \dots, n\}}$. This function is subject to the following linear constraints. (Recall that we use $f_{<t}$ to denote the common weight on the sets of size less than t .

This is identical to f_{t-1} .)

$$W(f) = w \quad (1)$$

$$0 \leq f_{<t} \quad (2)$$

$$f_i \leq f_{i+1} \text{ for all } i \in [t-1, n-1] \quad (3)$$

$$f_n \leq 1 \quad (4)$$

Let $R \subseteq [t-1, n] \times [t-1, n]$. Then we define P_R to be the set of $f \in \mathcal{C}_{n,t}^w$ subject to additional constraints

$$f_i + f_j \geq 1 \text{ if } (i, j) \in R \quad (5)$$

$$f_i + f_j \leq 1 \text{ if } (i, j) \notin R \quad (6)$$

Notice that given $f \in \mathcal{C}_{n,t}^w$, every pair $(i, j) \in [t-1, n] \times [t-1, n]$ has $f_i + f_j \leq 1$ or $f_i + f_j \geq 1$, and so f is in some P_R . More importantly, if we set

$$c_{i,j} = \begin{cases} \binom{n}{i} \Lambda(i, j) & \text{if } i \geq t; \\ \binom{n}{<t} \sum_{k=0}^{t-1} \Lambda(i, k) & \text{if } i = t-1, \end{cases}$$

then for all $f \in P_R$,

$$D_t(f) = \sum_{(i,j) \in R} c_{i,j} (f_i + f_j - 1).$$

Thus D_t is affine on each P_R . Fix R . We are now ready to find the vertices of P_R . We may think of the coefficients on a constraint as a vector in $\mathbb{R}^{n-t+2}$. For example, the constraint $f_i \leq f_{i+1}$ is equivalent to $f_i - f_{i+1} \leq 0$ and so it corresponds to a vector of the form

$$(0, \dots, 0, 1, -1, 0, \dots, 0).$$

Similarly, the constraint $f_i + f_j \geq 1$ becomes a vector with ones in positions i and j , and 0's elsewhere. Since we are thinking of $\mathcal{C}_{n,t}^w$ as an $n-t+2$ dimensional space, a fractional set system is a vertex if it achieves equality for $n-t+2$ linearly independent constraints. Notice that equality always holds for constraint (1), and so we want equality to hold for $n-t+1$ linearly independent constraints of types (2)-(6). Let $f \in P_R$ be a vertex of P_R . Given $\delta \in [1/2, 1]$, define

$$S_\delta = \{i \in [t-1, n] : f_i = \delta \text{ or } f_i = 1 - \delta\}.$$

Define C_δ to be the set of coefficient vectors for the constraints of types (2)-(6) that f exactly meets, and where for some $i \in S_\delta$, the i^{th} entry is nonzero. Let C be the set of vectors corresponding to all constraints for which f achieves equality. By our choice of f , we have $\text{rank } C = n-t+2$. On the other hand, if $\vec{w}$ is the vector corresponding to the weight constraint, then

$$C = \{\vec{w}\} \cup \bigcup_{\delta \in [1/2, 1]} C_\delta,$$

and so

$$\text{rank } C \leq 1 + \sum_{\delta \in [1/2, 1]} \text{rank } C_\delta.$$

Notice that this sum is finite, since f only takes on finitely many values. Notice further that the only nonzero coefficients in a constraint in C_δ are on f_i with $f_i = \delta$ or $f_i = 1 - \delta$. Thus $\text{rank } C_\delta \leq |S_\delta|$. Consider δ not equal to $1/2$ or 1 . Given $v \in [0, 1]$, denote $f^{-1}(v) = \{i \in [t-1, n] : f_i = v\}$. Let $\vec{p} \in \mathbb{R}^{\{t-1, t, \dots, n\}}$ be the vector that is 1 on $f^{-1}(\delta)$, -1 on $f^{-1}(1 - \delta)$, and 0 everywhere else. Notice that no vector in C_δ corresponds to constraints (2) or (4), and so in particular for all $\vec{v} \in C_\delta$, we have $\vec{v} \cdot \vec{p} = 0$. Thus when we restrict our vectors to S_δ (the support of C_δ), the dimension of the space perpendicular to C_δ is at least 1. This implies that

$$\text{rank } C_\delta \leq |S_\delta| - 1$$

Thus if d is the number of distinct nonempty S_δ other than S_1 and $S_{1/2}$, we have

$$\begin{aligned} n - t + 2 &= \text{rank } C \\ &\leq 1 + \sum_{\delta \in [1/2, 1]} \text{rank } C_\delta \\ &\leq 1 - d + \sum_{\delta \in [1/2, 1]} |S_\delta| \\ &= 1 - d + n - t + 2 \end{aligned}$$

By canceling terms and rearranging, $d \leq 1$, and the claim is proved. $\square$

Proof of Theorem 14. First we see that $D_{n,t}$ is convex: let w_1 and w_2 have $0 \leq w_1 \leq w_2 \leq 2^n$. Then for $i = 1$ or 2 , there exists $f_i \in \mathcal{C}_{n,t}^{w_i}$ such that $D_{n,t}(w_i) = D_t(f_i)$. Let $\lambda \in [0, 1]$. Then

$$\begin{aligned} D_{n,t}(\lambda w_1 + (1 - \lambda)w_2) &\leq D_t(\lambda f_1 + (1 - \lambda)f_2) \\ &\leq \lambda D_t(f_1) + (1 - \lambda)D_t(f_2) \\ &= \lambda D_{n,t}(w_1) + (1 - \lambda)D_{n,t}(w_2) \end{aligned}$$

Next we see that $D_{n,t}$ is piecewise linear. Let $l_0, l_{1-\delta}, l_{1/2}, l_\delta, l_1$ be nonnegative integers that sum to $n + 1$. Also, let $\delta \in [1/2, 1]$. Define

$$f_\delta = (\underbrace{0, \dots, 0}_{l_0}, \underbrace{1 - \delta, \dots, 1 - \delta}_{l_{1-\delta}}, \underbrace{1/2, \dots, 1/2}_{l_{1/2}}, \underbrace{\delta, \dots, \delta}_{l_\delta}, \underbrace{1, \dots, 1}_{l_1})$$

Notice that the weight of f_δ is affine in δ , as is $D_{n,t}(f_\delta)$. Thus the graph of $\{(W(f_\delta), D_t(f_\delta)) : \delta \in [1/2, 1]\}$ is a line segment. By Lemma 15, $D_{n,t}(w)$ is the

minimum value among all the line segments of this type that are defined at w . Thus $D_{n,t}$ is piecewise linear.

Changes of slope occur at the end of the line segments described above or at the intersection of two of them. As it turns out, slope does not change at an intersection of two of these line segments if the intersection is not also an endpoint for one of the line segments. This is true because otherwise near the point of intersection, $D_{n,t}$ would be the minimum of two line segments, which is not a convex function. Thus the slope of $D_{n,t}$ changes at endpoints of the line segments described above. But the endpoints occur where $W(f_\delta)$ is maximized or minimized. These extrema occur when $\delta = 1/2$ and $\delta = 1$. In either case, f_δ is pseudo-ball. $\square$

6. Conclusion

In summary, this paper shows that the specific proof technique (via fractional set systems) used in [6] to establish the minimum number of disjoint pairs in a set system does not transfer to the problem of minimizing the number of t -disjoint pairs when $t \geq 2$. We provide some insight into the structure of extremal fractional set systems. The problems of determining extremal set systems and extremal fractional set systems are still open.

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